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Pair production in a cavity with moving boundaries

Jaume Haro

Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

E-mail: jaime.haro@upc.edu

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Abstract
We study the creation of charged scalar massive particles in a three-dimensional cavity with a moving wall. Using the Schrödinger picture, we calculate the average number of produced pairs, and the Casimir energy. We also discuss the divergent terms that appear in our calculations. Finally, we present a model that includes the back reaction of the field, and that may be used to remove the divergent terms.

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1. Introduction

The creation of photons in a cavity with moving boundaries has been studied in many papers. In some of them (see [8, 11, 15, 16, 18]), the authors study cavities whose walls oscillate in resonance with the eigenfrequencies of some electromagnetic modes. In this specific case, the authors calculate, using a non-perturbative way (multiple scale analysis [15], rotating wave approximation [5, 16]) the average number of produced photons in every mode. In other papers (see [1–3, 5, 12]), the authors study the creation of massless particles in cavities with walls moving in a more general prescribed trajectory. In this case, due to the difficulty of the problem, the number of produced pairs is only computed until the first or second perturbative order.

In this work, we also study this second case, but from another point of view. We consider a rectangular cavity with a moving wall, and we assume that the velocity of the wall is of order ϵ. Then following [9], we make a change of coordinates that transform the moving wall into a stationary one. Then, using these coordinates we can quantize any field in the usual way. Here, to simplify, we consider a charged massive scalar field, although the method is also valid for the Dirac or the electromagnetic field. Once we have quantized the field, we calculate the time-evolved vacuum state in the same way as [12], and then we obtain, until order ϵ^2, the average number of produced pairs and the Casimir energy. In the obtained formulae, there appear some divergent terms when the movement of the wall is not smooth enough [1]. We believe that these divergent quantities can be removed taking into account the back reaction. For this reason, using the new coordinates, we easily construct the Lagrangian of the system that includes the back reaction (this Lagrangian coincides with that obtained in [6] following another approach), and finally we present a simplified model that may be used to remove the divergent terms.

2. The (1+1)-dimensional case

2.1. The problem

In this section, we consider a one-dimensional cavity, with a stationary boundary located at the origin, and the other moving in a prescribed trajectory, denoted by $l_\epsilon(t) = L + \epsilon f(t)$.

We assume that ϵ is a dimensionless small parameter, and we also suppose that the function f has the following form,

$$f(t) = \begin{cases} 0 & \text{when} \quad t < 0 \\ f(t) & \text{when} \quad 0 \leqslant t \leqslant T \\ \bar{L} & \text{when} \quad t > T, \end{cases} \tag{1}$$

that is, the movement of the boundary starts at time 0 and finishes at time T.

The Lagrangian density of the charged massive scalar field is

$$\mathcal{L}(t,x) = \hbar^2|\phi_t|^2 - c^2\hbar^2|\phi_x|^2 - m^2c^4|\phi|^2, \tag{2}$$

where we must impose Dirichlet boundary conditions, i.e., we impose that ϕ satisfies $\phi(t,0) = \phi(t, l_\epsilon(t)) = 0$, $\forall t \in \mathbb{R}$.

To transform the moving boundary into a fixed one, we make the following *not conformal* change of coordinates (see [9]):

$$\mathcal{R} : (s,u) \to (t(s,u), x(s,u)) = (s, ul_\epsilon(s)). \tag{3}$$

Now, the boundaries are situated at the points $u = 0$ and $u = 1$, and the Lagrangian density of the system behaves as $\tilde{\mathcal{L}}(s,u) := \mathcal{L}(\mathcal{R}(s,u))l_\epsilon(s)$.

If we write $\tilde{\phi}(s,u) := \phi(\mathcal{R}(s,u))$, we can get the Lagrangian density as a function of the new coordinates

$$\tilde{\mathcal{L}} = l_\epsilon\left[\hbar^2\left|\tilde{\phi}_s - \frac{l'_\epsilon}{l_\epsilon}u\tilde{\phi}_u\right|^2 - \frac{c^2\hbar^2}{l_\epsilon^2}|\tilde{\phi}_u|^2 - m^2c^4|\tilde{\phi}|^2\right], \tag{4}$$

with boundary condition $\tilde{\phi}(s,0) = \tilde{\phi}(s,1) = 0$.

The Hamiltonian density is

$$\tilde{\mathcal{H}} = \frac{1}{l_\epsilon}\left[\frac{|\tilde{\xi}|^2}{\hbar^2} + c^2\hbar^2|\tilde{\phi}_u|^2 + l_\epsilon^2m^2c^4|\tilde{\phi}|^2\right] + \frac{l'_\epsilon}{l_\epsilon}u(\tilde{\phi}_u^*\tilde{\xi} + \tilde{\xi}^*\tilde{\phi}_u), \tag{5}$$

where we have used the canonical conjugated momentum

$$\tilde{\xi} := \frac{\partial\tilde{\mathcal{L}}}{\partial\tilde{\phi}_s^*} = l_\epsilon\hbar^2\left(\tilde{\phi}_s - \frac{l'_\epsilon}{l_\epsilon}u\tilde{\phi}_u\right). \tag{6}$$

2.2. Decomposition in oscillators

To obtain the decomposition in harmonic oscillators of the Hamiltonian, we start with the Fourier series of the dynamical variables $\tilde{\phi}$ and $\tilde{\xi}$:

$$\tilde{\phi}(s,u) = \sum_{n=1}^{\infty} A_n(s)\sqrt{2}\sin(n\pi u), \qquad \tilde{\xi}(s,u) = \sum_{n=1}^{\infty} B_n(s)\sqrt{2}\sin(n\pi u).$$

It is easy to check that the Hamiltonian has the form

$$\tilde{H} = \frac{1}{l_\epsilon}\sum_{n=1}^{\infty}\left[\frac{|B_n|^2}{\hbar^2} + \left(c^2\hbar^2\pi^2n^2 + l_\epsilon^2m^2c^4\right)|A_n|^2\right] - \frac{l'_\epsilon}{2l_\epsilon}\sum_{n=1}^{\infty}(B_nA_n^* + B_n^*A_n)$$
$$- \frac{l'_\epsilon}{l_\epsilon}\sum_{\substack{r,k=1\\ r\neq k}}^{\infty}\frac{2kr}{r^2-k^2}(-1)^{r+k}(B_rA_k^* + B_r^*A_k), \tag{7}$$

where, now the dynamical variables are $A_n, \ldots, B_n^*$, with $n \in \mathbb{N}$.

If we want to obtain the decomposition in harmonic oscillators of the Hamiltonian for times smaller than zero, we must make the canonical change

$$A_n = \frac{1}{\hbar\sqrt{2L}}(Q_n + \mathrm{i}\bar{Q}_n), \qquad B_n = \hbar\sqrt{\frac{L}{2}}(P_n + \mathrm{i}\bar{P}_n),$$

where Q_n and $\bar{Q}_n$ are two independent real variables and, P_n and $\bar{P}_n$ are their canonically conjugated momenta.

Then, $\forall s \leqslant 0$ we have

$$\tilde{H}(s) = \frac{1}{2}\sum_{n=1}^{\infty}\left(P_n^2 + \omega_n^2(0)Q_n^2\right) + \left(\bar{P}_n^2 + \omega_n^2(0)\bar{Q}_n^2\right), \tag{8}$$

where the frequencies of the oscillators are given by $\omega_n(0) \equiv \frac{1}{\hbar}\sqrt{\frac{c^2\hbar^2\pi^2n^2}{L^2} + m^2c^4}$.

To obtain this decomposition for times greater than T, we must make this other canonical change

$$A_n = \frac{1}{\hbar\sqrt{2(L+\epsilon\bar{L})}}(q_n + \mathrm{i}\bar{q}_n), \qquad B_n = \hbar\sqrt{\frac{L+\epsilon\bar{L}}{2}}(p_n + \mathrm{i}\bar{p}_n),$$

and then, $\forall s \geqslant T$ we have

$$\tilde{H}(s) = \frac{1}{2}\sum_{n=1}^{\infty}\left(p_n^2 + \omega_n^2(T)q_n^2\right) + \left(\bar{p}_n^2 + \omega_n^2(T)\bar{q}_n^2\right), \tag{9}$$

where now the frequencies are given by $\omega_n(T) \equiv \frac{1}{\hbar}\sqrt{\frac{c^2\hbar^2\pi^2n^2}{(L+\epsilon\bar{L})^2} + m^2c^4}$.

When the boundary moves, that is, for $s \in (0, T)$, the Hamiltonian has a very complicated form. Here, using the dynamical variables $Q_n, \ldots, \bar{P}_n$ we can get the following expression,

$$\tilde{H}(s) = \frac{L}{2l_\epsilon}\sum_{n=1}^{\infty}\left(P_n^2 + \lambda_n^2(\epsilon; s)Q_n^2\right) + \left(\bar{P}_n^2 + \lambda_n^2(\epsilon; s)\bar{Q}_n^2\right) - \frac{l'_\epsilon}{2l_\epsilon}\sum_{n=1}^{\infty}(Q_nP_n + \bar{Q}_n\bar{P}_n)$$
$$- \frac{l'_\epsilon}{l_\epsilon}\sum_{\substack{r,k=1\\ r\neq k}}^{\infty}\frac{2kr}{r^2-k^2}(-1)^{r+k}(Q_kP_r + \bar{Q}_k\bar{P}_r); \tag{10}$$

where

$$\lambda_n(\epsilon; s) = \frac{1}{\hbar}\sqrt{\frac{c^2\hbar^2\pi^2n^2}{L^2} + \frac{l_\epsilon^2}{L^2}m^2c^4}.$$

2.3. Quantum theory

The quantum theory, in the Schrödinger picture, is obtained imposing the standard commutation rules:

$$[\hat{P}_r, \hat{Q}_l] = [\hat{\bar{P}}_r, \hat{\bar{Q}}_l] = -\mathrm{i}\hbar\delta_{r,l}, \qquad [\hat{P}_r, \hat{\bar{Q}}_l] = [\hat{\bar{P}}_r, \hat{Q}_l] = [\hat{P}_r, \hat{\bar{P}}_l] = [\hat{Q}_r, \hat{\bar{Q}}_l] = 0.$$

Remark 2.1. The same rules are valid for $\hat{q}_k, \ldots, \hat{\bar{p}}_k$.

Now, using these operators we can define, in the Schrödinger picture, the creation and the annihilation operators [17].

For $s \leqslant 0$, the particle and antiparticle annihilation operators are, respectively,

$$\hat{a}_n = \frac{(\mathrm{i}\hat{P}_n + \omega_n(0)\hat{Q}_n) + \mathrm{i}(\mathrm{i}\hat{\bar{P}}_n + \omega_n(0)\hat{\bar{Q}}_n)}{2\sqrt{\hbar\omega_n(0)}}, \qquad \hat{b}_n = \frac{(\mathrm{i}\hat{P}_n + \omega_n(0)\hat{Q}_n) - \mathrm{i}(\mathrm{i}\hat{\bar{P}}_n + \omega_n(0)\hat{\bar{Q}}_n)}{2\sqrt{\hbar\omega_n(0)}}.$$

For $s \geqslant T$, the particle and antiparticle annihilation operators are, respectively,

$$\hat{c}_n = \frac{(\mathrm{i}\hat{p}_n + \omega_n(T)\hat{q}_n) + \mathrm{i}(\mathrm{i}\hat{\bar{p}}_n + \omega_n(T)\hat{\bar{q}}_n)}{2\sqrt{\hbar\omega_n(T)}}, \qquad \hat{d}_n = \frac{(\mathrm{i}\hat{p}_n + \omega_n(T)\hat{q}_n) - \mathrm{i}(\mathrm{i}\hat{\bar{p}}_n + \omega_n(T)\hat{\bar{q}}_n)}{2\sqrt{\hbar\omega_n(T)}}.$$

From the definition of these operators, we can easily deduce the relation between the operators at time T and the operators at time 0:

$$\hat{c}_n = \hat{a}_n + \frac{\epsilon\bar{L}}{2L}\frac{m^2c^4}{\hbar^2\omega_n^2(0)}\hat{b}_n^\dagger + \mathcal{O}(\epsilon^2), \qquad \hat{d}_n^\dagger = \hat{b}_n^\dagger + \frac{\epsilon\bar{L}}{2L}\frac{m^2c^4}{\hbar^2\omega_n^2(0)}\hat{a}_n + \mathcal{O}(\epsilon^2). \tag{11}$$

Using the creation and annihilation operators, we easily check that

$$\hat{H}(s) = \sum_{n=1}^{\infty} \hbar\omega_n(0)\big(\hat{a}_n^\dagger\hat{a}_n + \hat{b}_n^\dagger\hat{b}_n + 1\big) \quad \forall s \leqslant 0.$$

$$\hat{\hat{H}}(s) = \sum_{n=1}^{\infty} \hbar\omega_n(T)\big(\hat{c}_n^\dagger\hat{c}_n + \hat{d}_n^\dagger\hat{d}_n + 1\big) \quad \forall s \geqslant T.$$

When the boundary moves, the Hamiltonian is what appears in equation (10). Then to find a self-adjoint quantum Hamiltonian, we must use the symmetrization rule

$$Q_nP_n + \bar{Q}_n\bar{P}_n \longrightarrow \tfrac{1}{2}(\hat{Q}_n\hat{P}_n + \hat{P}_n\hat{Q}_n + \hat{\bar{Q}}_n\hat{\bar{P}}_n + \hat{\bar{P}}_n\hat{\bar{Q}}_n).$$

In this case, if we use the operators at time 0, the Hamiltonian has a very complicated form (see, for details, [3, 5]).

2.4. The average number of produced pairs

Let $\mathcal{T}^s$ be the quantum evolution operator of the Schrödinger equation, and let $\mathcal{T}_0^s$ be the 'free evolution operator', that is, the evolution operator of the equation

$$\mathrm{i}\hbar\frac{\partial}{\partial s}|\psi\rangle = \hat{\hat{H}}(0)|\psi\rangle. \tag{12}$$

The well-known relation between these two operators is

$$\mathcal{T}^s = \mathcal{T}_0^s - \frac{\mathrm{i}}{\hbar}\int_0^s \mathcal{T}_0^{s-\tau}\hat{V}(\tau)\mathcal{T}^\tau\,\mathrm{d}\tau, \tag{13}$$

where $\hat{V}(\tau)$ is the part of the quantum Hamiltonian that depends on the parameter ϵ.

Then, if we apply the Picard method to relation (13), we can obtain the second-order approximation of the Schrödinger evolution operator (see [12])

$$\mathcal{T}^s \sim \mathcal{T}_0^s - \frac{\mathrm{i}}{\hbar}\int_0^s \mathcal{T}_0^{s-\tau}\hat{V}(\tau)\mathcal{T}_0^{\tau}\,\mathrm{d}\tau - \frac{1}{\hbar^2}\int_0^s\int_0^\tau \mathcal{T}_0^{s-\tau}\hat{V}(\tau)\mathcal{T}_0^{\tau-\mu}\hat{V}(\mu)\mathcal{T}_0^{\mu}\,\mathrm{d}\mu\,\mathrm{d}\tau. \tag{14}$$

Now, let $|0\rangle$ be the initial vacuum state. Then, applying the evolution operator to this state, we can conclude that the average number of produced pairs is

$$\begin{aligned} N_m &:= \sum_{n=1}^{\infty}\langle 0|(\mathcal{T}^T)^\dagger \hat{c}_n^\dagger \hat{c}_n \mathcal{T}^T|0\rangle \\ &= \frac{\epsilon^2}{L^2}\left(\frac{c\pi}{L}\right)^4 \sum_{k,n=1}^{\infty}\frac{k^2n^2\left|\int_0^T f'(\tau)\,\mathrm{e}^{\mathrm{i}(\omega_n(0)+\omega_k(0))\tau}\,\mathrm{d}\tau\right|^2}{\omega_n(0)\omega_k(0)(\omega_n(0)+\omega_k(0))^2} + \mathcal{O}(\epsilon^4), \end{aligned} \tag{15}$$

where m denotes the mass of the field.

Remark 2.2. From this formula, if we integrate by parts, we can easily deduce that, when the velocity of the boundary is discontinuous, the average of produced pairs is infinite. And when the velocity is continuous, this number is finite.

For massless particles, it is not difficult to check the formula

$$N_0 = \sum_{n=1}^{\infty}\langle 0|(\mathcal{T}^T)^\dagger \hat{c}_n^\dagger \hat{c}_n \mathcal{T}^T|0\rangle = \frac{\epsilon^2}{6L^2}\sum_{n=1}^{\infty}\left(n-\frac{1}{n}\right)\left|\int_0^T f'(\tau)\,\mathrm{e}^{\mathrm{i}\frac{c\pi}{L}\tau n}\,\mathrm{d}\tau\right|^2 + \mathcal{O}(\epsilon^4). \tag{16}$$

If f has the form $f(t) = lg(\nu_1 t, \ldots, \nu_N t)$, where g is a dimensionless function, l is a distance and $\nu_1, \ldots, \nu_N$ are frequencies. Then, from the formula (16), if we suppose that f'' has a discontinuity at $t_0 \in [0, T]$, and assuming that $\frac{L\nu_k}{c} \ll 1$ for $k = 1, \ldots, N$, we obtain the same result as Moore's (see [1]),

$$N_0 \approx \frac{\epsilon^2 L^2}{6(c\pi)^4}\left(f''\left(t_0^+\right) - f''\left(t_0^-\right)\right)^2(\zeta_R(3) - \zeta_R(5)). \tag{17}$$

Remark 2.3. If we consider the trajectory $l_\epsilon(t) = L\left(1+\epsilon\sin\left(\frac{2c\pi}{L}t\right)\right)$, and we take $T_N = \frac{2L}{c}N$ with $N \in \mathbb{N}$. Using the formula (16), we obtain exactly

$$N_0 = \frac{\epsilon^2}{4}\left(\frac{c\pi}{L}T_N\right)^2 + \mathcal{O}(\epsilon^4).$$

This expression coincides with the result of [8, 11].

3. The (3+1)-dimensional case

3.1. The average number of produced pairs

Here, we consider a rectangular cavity with a moving wall, that is, a volume of the form

$$[0, l_\epsilon(t)] \times [0, l_2] \times [0, l_3].$$

To obtain the number of created pairs in the (3+1)-dimensional case, we make the substitution $m^2c^4 \to \frac{\hbar^2c^2\pi^2}{l_2^2}i^2 + \frac{\hbar^2c^2\pi^2}{l_3^2}j^2 + m^2c^4$ in the formula (15), and we take the sum over i and j. Then, if we suppose that $l_2, l_3 \gg L$, until order ϵ^2, we have

$$N_m(L) \approx \frac{\epsilon^2}{4L^2}\frac{l_2 l_3}{(c\pi\hbar)^2}\left(\frac{c\pi}{L}\right)^4\sum_{k,n=1}^{\infty}\int_{\mathbb{R}^2}\frac{k^2n^2\left|\int_0^T f'(\tau)\,\mathrm{e}^{\mathrm{i}(A(k)+A(n))\tau}\,\mathrm{d}\tau\right|^2}{A(k)A(n)(A(k)+A(n))^2}\,\mathrm{d}y\,\mathrm{d}z, \tag{18}$$

where $A(n) \equiv \frac{1}{\hbar}\sqrt{\left(\frac{c\pi\hbar n}{L}\right)^2 + y^2 + z^2 + m^2c^4}$.

Note that, in the 3+1 case, if we integrate by parts the formula (18), we can see that, when the velocity of the wall is continuous and the acceleration is discontinuous, the number of produced pairs is *infinite*. We can also see that when f'' is continuous the average number of created pairs is finite.

3.2. The Casimir energy

Here, we calculate the Casimir energy when the movement of the wall is finished, i.e., when $s > T$.

In this situation, if we assume that $l_2, l_3 \gg L$, the energy of the system is

$$\begin{aligned} E_m(L;T) &= \langle 0|(\mathcal{T}^T)^\dagger \hat{\hat{H}}(T)\mathcal{T}^T|0\rangle \\ &= \frac{\epsilon^2}{4L^2}\frac{l_2 l_3}{(c\pi\hbar)^2}\left(\frac{c\pi}{L}\right)^4 \sum_{k,n=1}^{\infty}\int_{\mathbb{R}^2}\frac{k^2n^2\left|\int_0^T f'(\tau)\,\mathrm{e}^{\mathrm{i}(A(k)+A(n))\tau}\,\mathrm{d}\tau\right|^2}{A(k)A(n)(A(k)+A(n))}\,\mathrm{d}y\,\mathrm{d}z \\ &\quad + \text{Terms until order } \epsilon^2 \text{ of } \left(\frac{1}{4}\frac{l_1 l_2\hbar}{(c\pi\hbar)^2}\sum_{n=1}^{\infty}\int_{\mathbb{R}^2} A(n;T)\,\mathrm{d}y\,\mathrm{d}z\right) + \mathcal{O}(\epsilon^4), \end{aligned} \tag{19}$$

where $A(n;T) \equiv \frac{1}{\hbar}\sqrt{\left(\frac{c\pi\hbar n}{L+\epsilon L}\right)^2 + y^2 + z^2 + m^2c^4}$.

Note that, the energy is decomposed in two different parts. The first, which we call the dynamical part, is the energy of the produced pairs, and the other, which we call the static part, is the usual Casimir energy that appears is the stationary case.

It is clear that the static part is divergent. Then, to renormalize this part we subtract the energy of the Minkowskian vacuum inside the box, namely $E_m^{\mathrm{Min}}(L)$ [14]. Then, the new energy of the system is

$$\mathcal{E}_m(L) \equiv E_m(L) - E_m^{\mathrm{Min}}(L). \tag{20}$$

Now, this new energy is infinite in the case when the acceleration of the wall is discontinuous. But, from the Abel–Plana formula [10] we can deduce that the divergent part of $\mathcal{E}_m(L)$ does not depend on the length L, and consequently the Casimir force

$$F_m(L) \equiv -\frac{\mathrm{d}}{\mathrm{d}L}\mathcal{E}_m(L), \tag{21}$$

is a finite quantity.

3.3. The back reaction

Here we also consider a rectangular cavity with a moving wall. Let $l(s) = l_\epsilon(s) + g(s)$ be the trajectory of the boundary, where l_ϵ is the prescribed trajectory of the wall, that is, l_ϵ is the solution of the Newton equation $M\ddot{x} = F_{\mathrm{ext}}$ where F_{ext} is the prescribed external force, and g describes the correction of the movement produced by the emission of pairs. Then, the full Lagrangian of the system is

$$\begin{aligned} \tilde{L} = \int_0^{l_3}\int_0^{l_2}\int_0^1 l\Bigg[\hbar^2\left|\tilde{\phi}_s - \frac{l'}{l}u\tilde{\phi}_u\right|^2 - \frac{c^2\hbar^2}{l^2}|\tilde{\phi}_u|^2 \\ - c^2\hbar^2(|\tilde{\phi}_y|^2 + |\tilde{\phi}_z|^2) - m^2c^4|\tilde{\phi}|^2\Bigg]\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{2}M(l'(s))^2 - W(l(s),s). \end{aligned} \tag{22}$$

The first term is the Lagrangian of the field in terms of the coordinates (s, u, y, z) introduced in section 1.1, and the other terms are the kinetic and potential energy of the moving boundary.

The full Hamiltonian analogous to that obtained in [6] is

$$\tilde{H}_{\text{full}} = \frac{1}{l}\int_0^{l_3}\int_0^{l_2}\int_0^1 \left[\frac{|\tilde{\xi}|^2}{\hbar^2} + c^2\hbar^2|\tilde{\phi}_u|^2 + c^2\hbar^2 l^2(|\tilde{\phi}_y|^2 + |\tilde{\phi}_z|^2) + l^2m^2c^4|\tilde{\phi}|^2\right]\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}z$$
$$+ \frac{l'_\epsilon}{l}\int_0^{l_3}\int_0^{l_2}\int_0^1 u(\tilde{\phi}_u^*\tilde{\xi} + \tilde{\xi}^*\tilde{\phi}_u)\,\mathrm{d}u\,\mathrm{d}y\,\mathrm{d}z + \frac{1}{2}M((g')^2 - (l'_\epsilon)^2) + W(l(s),s). \tag{23}$$

To obtain the quantum theory, we must quantize the dynamical variables $\tilde{\phi}, \tilde{\xi}, g$ and $p \equiv \frac{\partial \bar{L}}{\partial g'}$ [6, 7]. But, we believe that to simplify the calculation of the average number of produced pairs and the Casimir energy (taking into account the back reaction), it is easier to use the following approximated model.

From the Euler–Lagrange equations, we obtain the following dynamical equation:

$$g'' = F(\tilde{\phi}, \ldots, \tilde{\xi}^*, g', g, l_\epsilon).$$

Now we only quantize the variables $\tilde{\phi}, \ldots, \tilde{\xi}^*$, and we impose the equation

$$g''(s) = \langle 0|(\mathcal{T}^s)^\dagger F(\hat{\tilde{\phi}}, \ldots, \hat{\tilde{\xi}}^*, g', g, l_\epsilon)\mathcal{T}^s|0\rangle \equiv \mathcal{F}(s).$$

Then, the model that includes the back reaction is

$$\begin{cases} \mathrm{i}\hbar\partial_s\mathcal{T}^s|0\rangle = \hat{\tilde{H}}_{\text{full}}(s)\mathcal{T}^s|0\rangle \\ g''(s) = \mathcal{F}(s). \end{cases} \tag{24}$$

3.4. Open questions

(i) In dimension 3, when f'' has a discontinuity the average number of produced pairs is infinite. Why is this number infinite? Is it possible to renormalize this quantity?

(ii) For a rectangular cavity $[0, l_\epsilon(s)] \times [0, l_2] \times [0, l_3]$ an easy but cumbersome calculation shows that the dynamical part of the energy at time s is given by

$$\frac{\epsilon^2\hbar}{L^2}\left(\frac{\pi c}{L}\right)^4 \sum_{\mathbf{n},\mathbf{k}\in\mathbb{N}^3} \frac{n_1^2k_1^2\delta_{n_2,k_2}\delta_{n_3,k_3}}{\omega_{\mathbf{n}}(0)\omega_{\mathbf{k}}(0)(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))^3}$$
$$\times\left[\left|\int_0^s f''(\tau)\,\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))\tau}\mathrm{d}\tau\right|^2 - (f'(s))^2\right] - \frac{\epsilon^2}{2L^2\hbar}\left(\frac{\pi c}{L}\right)^2$$
$$\times\sum_{\mathbf{n}\in\mathbb{N}^3}\frac{n_1^2A_{\mathbf{n}}^2}{\omega_{\mathbf{n}}^4(0)} f'(s)\int_0^2 f'(\tau)\sin(2\omega_{\mathbf{n}}(0)(s-\tau))\,\mathrm{d}\tau + \mathcal{O}(\epsilon^4), \tag{25}$$

where we have introduced

$$\omega_{\mathbf{n}}(0) \equiv \frac{1}{\hbar}\sqrt{\left(\frac{\pi\hbar c n_1}{L}\right)^2 + \left(\frac{\pi\hbar c n_2}{l_2}\right)^2 + \left(\frac{\pi\hbar c n_3}{l_3}\right)^2 + m^2c^4}$$

and

$$A_{\mathbf{n}} \equiv \sqrt{\left(\frac{\pi\hbar c n_2}{l_2}\right)^2 + \left(\frac{\pi\hbar c n_3}{l_3}\right)^2 + m^2c^4}.$$

Note that, this dynamical energy has some divergent terms. If we assume that $l_2, l_3 \gg L$, using the Abel–Plana formula (see (2.29)–(2.33) of [10]), we can conclude that the divergent part is

$$\frac{\epsilon^2 l_2 l_3}{(\pi^2 c^2 \hbar)^2}\int_{\mathbb{R}^4}\frac{x^2\bar{x}^2\,\mathrm{d}x\,\mathrm{d}\bar{x}\,\mathrm{d}y\,\mathrm{d}z}{E_{xyz}E_{\bar{x}yz}(E_{xyz}+E_{\bar{x}yz})^3}\left[\frac{\hbar^2(f''(s))^2}{(E_{xyz}+E_{\bar{x}yz})^2}-(f'(s))^2\right]$$
$$-\frac{\epsilon^2\hbar}{L}\frac{l_2 l_3}{(\pi c\hbar)^2}\int_{\mathbb{R}^3}\frac{x^2 A_{yz}^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{E_{xyz}^4}\left[\frac{\hbar(f'(s))^2}{2E_{xyz}}-\frac{\hbar^3 f'(s)f'''(s)}{8E_{xyz}^3}\right], \tag{26}$$

where

$$E_{xyz}\equiv\sqrt{x^2+y^2+z^2+m^2c^4}, \qquad A_{yz}\equiv\sqrt{y^2+z^2+m^2c^4}.$$

We write (26) in the following form,

$$C_1(f'(s))^2+C_2(f''(s))^2+C_3 f'(s)f'''(s), \tag{27}$$

where C_1, C_2 and C_3 are divergent constants.

C_1 can be removed by a mass renormalization of the moving wall [4]. To remove the other divergent constants we can follow a method similar to that used in [7]. We propose that the potential energy of the moving wall has the form

$$W(l_\epsilon(s),s)=W_{\mathrm{exp}}(l_\epsilon(s),s)-C_2(f''(s))^2-C_3 f'(s)f'''(s). \tag{28}$$

where W_{exp} is the experimental potential energy of the moving wall. Thus, the full energy of the system does not contain any divergent quantity.

We believe that (28) can be proved if we take into account the back reaction. In fact, if we take into account the back reaction, the potential energy of the moving wall has the form

$$V+b(f''(s))^2+cf'(s)f'''(s), \tag{29}$$

and perhaps, using the model proposed in equation (24), we can show that

$$V=W_{\mathrm{exp}}, \qquad b=-C_2 \qquad \text{and} \qquad c=-C_3.$$

These questions will be discussed in [19].

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